

GEOMETRIC RTT REALIZATION OF $U_v(gl_n)^+$

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ABSTRACT. In this paper, we give a BLM realization of the positive part of the quantum group of $U_v(gl_n)$ with respect to RTT relations.

1. INTRODUCTION

There are two ways to study quantum groups. One is algebraic, the another is geometric.

In algebraic way, there are two methods. The first was adopted by Drinfeld [10, 12] and Jimbo [15] to define the quantum enveloping algebra $U_q(\mathfrak{g})$ as a q -deformation of the enveloping algebra $U(\mathfrak{g})$ in terms of the Chevalley generators and Serre relations based on the data coming from the corresponding Cartan matrix. The second approach to realize the quantum groups was through RTT method. The approach was from the quantum inverse scattering method developed by the Leningrad school. Sometimes it was more available for us. For example, In [13] Faddeev, Reshetikhin and Takhtajan have shown that both the quantum enveloping algebras $U_q(\mathfrak{g})$ and the dual quantum groups for finite classical simple Lie algebras \mathfrak{g} can be studied in the RTT method using the solutions R of the Yang-Baxter equation:

$$(1) \quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

In geometric way, there are many results on the geometric realization of quantum groups with respect to the Chevalley generators. For example, the \mathbb{C} -valued GL_n equivalence functions on the flag variety associated to the finite dimensional vector space over \mathbb{F}_q was considered by Beilinson, Lusztig, and McPherson. It gave realization of Schur algebra of $U_v(gl_n)$, and $U_v(gl_n)$ was the limit algebra of the Schur algebra. Also Du and Gu used the same way give the realization of $U_v(gl(m|n))$ [5]. Moreover, they also obtained a new basis containing the standard generators for quantum linear super group and explicit multiplication formulas between the generators and an arbitrary basis element. In affine case, Fu gave a BLM realization for $U_{\mathbb{Z}}(\widehat{gl}_n)$ [4]. By the similar way, Bao-Wang [2] and Fan-Li[3] gave the several new quantum groups and the Schur-like duality of type B/C and D. But there are few results on the geometric RTT realization. So the questions are if the the geometric realization of quantum groups with respect to the RTT relations can be given and if more information about the representation of quantum group can be given though the geometry RTT realization. In this paper, we use the BLM's way to construct the RTT realization of $U_v(gl_n)^+$.

The paper is organized as follows. In section 2, we recall the basic results on flag varieties and the RTT realization of $U_v(gl_n)$. In section 3, we calculate generator realizations of the Schur algebra, find a subalgebra of the limit algebra of the Schur algebra and show that it is isomorphic to the positive part of $U_v(gl_n)$.

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2. PRELIMINARY

In this section, let us recall some basic facts on flag varieties appear in [1] and the RTT realization of $U_v(gl_n)$.

Let \mathbb{F}_q be a finite field of q elements and of odd characteristic. For positive integers d and n , consider the set \mathcal{X} of n -step flags $V = (V_i)_{0 \leq i \leq n}$ in \mathbb{F}_q^d such that $V_0 = 0$, $V_i \subseteq V_{i+1}$.

Let $G = GL(\mathbb{F}_q^d)$, and G acts naturally on set \mathcal{X} . Let G act diagonally on the product $\mathcal{X} \times \mathcal{X}$. Set

$$(2) \quad \mathcal{A} = \mathbb{Z}[v^{\pm 1}].$$

Let $\mathcal{S}_{\mathcal{X}} = \mathcal{A}(\mathcal{X} \times \mathcal{X})^G$ be the set of all \mathcal{A} -valued G -invariant functions on $\mathcal{X} \times \mathcal{X}$. Clearly, the set $\mathcal{S}_{\mathcal{X}}$ is a free \mathcal{A} -module. Moreover, $\mathcal{S}_{\mathcal{X}}$ admits an associative \mathcal{A} -algebra structure ‘ $*$ ’ under a standard convolution product as discussed in [1]. In particular, when v is specialized to \sqrt{q} , we have

$$(3) \quad f * g(V, V') = \sum_{V'' \in \mathcal{X}} f(V, V'')g(V'', V'), \quad \forall V, V' \in \mathcal{X}.$$

Let us describe the G -orbits on $\mathcal{X} \times \mathcal{X}$. We start by introducing the following notations associated to a matrix $M = (m_{ij})_{1 \leq i, j \leq c}$.

$$(4) \quad \begin{aligned} \text{ro}(M) &= \left(\sum_{j=1}^c m_{ij} \right)_{1 \leq i \leq c}, \\ \text{co}(M) &= \left(\sum_{i=1}^c m_{ij} \right)_{1 \leq j \leq c}. \end{aligned}$$

We also write $\text{ro}(M)_i$ and $\text{co}(M)_j$ for the i -th and j -th component of the row vectors of $\text{ro}(M)$ and $\text{co}(M)$, respectively. For any pair (V, V') of flags in \mathcal{X} , we can assign an n by n matrix whose (i, j) -entry equal to $\dim \frac{V_{i-1} + V_i \cap V'_j}{V_{i-1} + V_i \cap V'_{j-1}}$. We have the following bijection.

$$(5) \quad G \backslash \mathcal{X} \times \mathcal{X} \simeq \Theta_d,$$

where Θ_d is the set of all matrices Θ_d in $\text{Mat}_{n \times n}(\mathbb{N})$ such that $\sum_{i,j} (\Theta_d)_{i,j} = d$

Definition 2.0.1. The algebra $U_v(gl_n)$ is generated by the elements t_{ij} and $\overline{t_{ij}}$ with $1 \leq i, j \leq n$ subject to the relations

$$\begin{aligned} t_{ij} &= \overline{t_{ij}} = 0, \quad 1 \leq i < j \leq n. \\ v^{\delta_{ij}} t_{ia} t_{jb} - v^{\delta_{ab}} t_{jb} t_{ia} &= (v - v^{-1})(\delta_{b < a} - \delta_{i < j}) t_{ja} t_{ib}, \\ v^{\delta_{ij}} \overline{t_{ia}} \overline{t_{jb}} - v^{\delta_{ab}} \overline{t_{jb}} \overline{t_{ia}} &= (v - v^{-1})(\delta_{b < a} - \delta_{i < j}) \overline{t_{ja}} \overline{t_{ib}}, \\ v^{\delta_{ij}} \overline{t_{ia}} t_{jb} - v^{\delta_{ab}} t_{jb} \overline{t_{ia}} &= (v - v^{-1})(\delta_{b < a} t_{ja} \overline{t_{ib}} - \delta_{i < j} \overline{t_{ja}} t_{ib}). \end{aligned}$$

3. BLM REALIZATION OF $U_v(gl_n)^+$ WITH RESPECT TO RTT RELATIONS

3.1. Calculus of the algebra \mathcal{S} . For simplicity, we shall denote \mathcal{S} instead of $\mathcal{S}_{\mathcal{X}}$. In this section, we determine the generators for \mathcal{S} and the associated multiplication formula.

Lemma 3.1.1. Assume V_1, V_2, V_3 are the vector space over F_q , $V_1 \stackrel{1}{\subset} V_2 \subset V_3$, and $\dim V_1 = n-1$, $\dim V_2 = n$, $\dim V_3 = m$. Set $S = \{V'_3 | V'_3 \stackrel{1}{\subset} V_3, V_1 \subset V'_3, V_2 \cap V'_3 \neq V_2\}$. Then

$$\#S = q^{m-n}.$$

Where $V_1 \stackrel{1}{\subset} V_2$ means $V_1 \subset V_2$ and $\dim V_2 - \dim V_1 = 1$.

$$\begin{aligned} \text{Proof. } \#S &= \#\{V'_3 | V'_3 \stackrel{1}{\subset} V_3, V_1 \subset V'_3\} - \#\{V'_3 | V'_3 \stackrel{1}{\subset} V_3, V_2 \subset V'_3\} \\ \#S &= \#\{V'_3 | V'_3 \stackrel{1}{\subset} V_3, V_1 \subset V'_3\} - \#\{V'_3 | V'_3 \stackrel{1}{\subset} V_3, V_2 \subset V'_3\} \\ &= \frac{q^{m-n+1}-1}{q-1} - \frac{q^{m-n}-1}{q-1} \\ &= q^{m-n}. \end{aligned}$$

□

Lemma 3.1.2. Assume V_1, V_2, V_3 are the vector space over F_q , $V_2 \subset V_3$, $V_1 \subset V_3$, $V_1 \cap V_2 = 0$, $\dim V_1 = 1$, $\dim V_2 = n$, $\dim V_3 = m$. Set $S = \{V'_3 | V'_3 \stackrel{1}{\subset} V_3, V_1 \cap V'_3 = 0, \dim V_2 \cap V'_3 = n-1\}$. Then

$$\#S = q^{m-1} - q^{m-n-1}.$$

$$\begin{aligned} \text{Proof. } \#S &= \#\{V' | V' \stackrel{1}{\subset} V_3\} - \#\{V' | V' \stackrel{1}{\subset} V_3, V_1 \subset V'\} - \#\{V' | V' \stackrel{1}{\subset} V_3, V_2 \subset V'\} + \#\{V' | V' \stackrel{1}{\subset} V_3, V_1 \subset V', V_2 \subset V'\} \\ &= \frac{q^m-1}{q-1} - \frac{q^{m-1}-1}{q-1} - \frac{q^{m-n}-1}{q-1} + \frac{q^{m-n-1}-1}{q-1} \\ &= q^{m-1} - q^{m-n-1}. \end{aligned}$$

□

Lemma 3.1.3. Let $A = (a_{ij}) \in \Theta_d$.

(a) Assume $B = (b_{ij}) \in \Theta_d$. There exist $1 \leq i_0 < i_1 \leq n$ such that $B - E_{i_0, i_1}$ is the diagonal matrices, and $\sum_i b_{ij} = \sum_k a_{jk}$. Then

$$e_B * e_A = \sum_{(\mathbf{j}, \mathbf{p})} f_{(\mathbf{j}, \mathbf{p})} e_{(A + \sum_{l=1}^m (E_{j_{l-1}, p_l} - E_{j_l, p_l}))},$$

where $(\mathbf{j}, \mathbf{p}) = ((j_1, p_1), \dots, (j_m, p_m))$ satisfied the conditions: $i_0 = j_0 < j_1 < \dots < j_m = i_1$, $1 \leq p_m < \dots < p_1 \leq n$, and for any $1 \leq k \leq m$, $a_{j_k, p_k} \geq 1$. $f_{(\mathbf{j}, \mathbf{p})} = f_1 f_2 \dots f_m$, where

$$f_l = \begin{cases} \frac{v^{2(1+\sum_{j \geq p_1} a_{j_0, j})} - v^{2\sum_{j > p_1} a_{j_0, j}}}{v^2 - 1} \prod_{k=j_0+1}^{k=j_1-1} v^{2\sum_{j \geq p_1} a_{k, j}} & \text{if } l = 1; \\ (v^{2\sum_{j \geq p_l} a_{j_{l-1}, j}} - v^{2(\sum_{j > p_l} a_{j_{l-1}, j-1})}) \prod_{k=j_{l-1}+1}^{k=j_l-1} v^{2\sum_{j \geq p_l} a_{k, j}} & \text{if } l > 1. \end{cases}$$

(b) Assume $C = (c_{ij}) \in \Theta_d$. There exist $n \geq i_0 > i_1 \geq 1$ such that $C - E_{i_0, i_1}$ is the diagonal matrices, and $\sum_i c_{ij} = \sum_k a_{jk}$. Then

$$e_C * e_A = \sum_{(\mathbf{j}, \mathbf{p})} f'_{(\mathbf{j}, \mathbf{p})} e_{(A + \sum_{l=1}^m (E_{j_{l-1}, p_l} - E_{j_l, p_l}))},$$

where $(\mathbf{j}, \mathbf{p}) = ((j_1, p_1), \dots, (j_m, p_m))$ satisfied the conditions: $i_0 = j_0 > j_1 > \dots > j_m = i_1$, $1 \leq p_1 < \dots < p_m \leq n$, and for any $1 \leq k \leq m$, $a_{j_k, p_k} \geq 1$. $f'_{(\mathbf{j}, \mathbf{p})} = f'_1 f'_2 \dots f'_m$, where

$$f'_l = \begin{cases} \frac{v^{2(1+\sum_{j \leq p_1} a_{j0,j})} - v^{2\sum_{j < p_1} a_{j0,j}}}{v^2 - 1} \prod_{k=j_1+1}^{k=j_0-1} v^{2\sum_{j \leq p_1} a_{k,j}} & \text{if } l = 1; \\ (v^{2\sum_{j \leq p_l} a_{j_{l-1},j}} - v^{2(\sum_{j < p_l} a_{j_{l-1},j-1})}) \prod_{k=j_l+1}^{k=j_{l-1}-1} v^{2\sum_{j \leq p_l} a_{k,j}} & \text{if } l > 1. \end{cases}$$

Proof. Assume $A' = A + \sum_{l=1}^m (E_{j_{l-1},p_l} - E_{j_l,p_l})$. Let $f = (V_1 \subset \cdots \subset V_{i_0-1} \subset V_{i_0} \subset \cdots \subset V_{i_1-1} \subset V_{i_1} \subset \cdots \subset V_n)$, $f' = (V'_1 \subset \cdots \subset V'_{i_0-1} \subset V'_{i_0} \subset \cdots \subset V'_{i_1-1} \subset V'_{i_1} \subset \cdots \subset V'_n)$ be such that $(f, f') \in \mathcal{O}_{A'}$. Set $V_0 = V'_0 = 0$.

To prove (a), we need to compute how many f'' such that $(f, f'') \in \mathcal{O}_B$, $(f'', f') \in \mathcal{O}_A$. Assume $f'' = (V''_1 \subset \cdots \subset V''_{i_0-1} \subset V''_{i_0} \subset \cdots \subset V''_{i_1-1} \subset V''_{i_1} \subset \cdots \subset V''_n)$. For any $i < i_0$ or $i \geq i_1$, $V''_i = V_i$.

First, we need to count how many V''_{i_0} there exist. Consider the set Z_{i_0} of all subspace U_{i_0} of V such that $V_{i_0-1} \subset U_{i_0} \subset V_{i_0}$, $V_{i_0-1} + V_{i_0} \cap V'_{p_1-1} \subset U_{i_0}$, and $(V_{i_0-1} + V_{i_0} \cap V'_{p_1}) \cap U_{i_0} \neq V_{i_0-1} + V_{i_0} \cap V'_{p_1}$.

$$\begin{aligned} \#Z_{i_0} &= \#\{U_{i_0} | U_{i_0} \supset V_{i_0-1} + V_{i_0} \cap V'_{p_1-1}\} - \#\{U_{i_0} | U_{i_0} \supset V_{i_0-1} + V_{i_0} \cap V'_{p_1}\} \\ &= \frac{v^{2(1+\sum_{j \geq p_1} a_{j0,j})} - v^{2\sum_{j > p_1} a_{j0,j}}}{v^2 - 1}. \end{aligned}$$

Second, we fix U_{i_0} . Consider the set Z_{i_0+1} of all subspace U_{i_0+1} of V such that $U_{i_0} \subset U_{i_0+1} \subset V_{i_0+1}$, $V_{i_0} \cap U_{i_0+1} = U_{i_0}$ (since $B_{i_0, i_0+1} = 0$), $U_{i_0} + V_{i_0+1} \cap V'_{p_1-1} \subset U_{i_0+1}$. That is, $V_{i_0+1} \cap V'_{p_1-1} + U_{i_0} \subset U_{i_0+1}$, $V_{i_0} \cap U_{i_0+1} = U_{i_0}$.

$$\begin{aligned} \#Z_{i_0+1} &= \#\{U_{i_0+1} | U_{i_0+1} \supset V_{i_0+1} \cap V'_{p_1-1} + U_{i_0}\} - \#\{U_{i_0+1} | U_{i_0+1} \supset V_{i_0} + V_{i_0+1} \cap V'_{p_1-1}\} \\ &= \frac{v^{2(1+\sum_{j \geq p_1} a_{j0+1,j})} - v^{2\sum_{j \geq p_1} a_{j0+1,j}}}{v^2 - 1} \\ &= v^{2\sum_{j \geq p_1} a_{j0+1,j}}. \end{aligned}$$

Similarly, as above, for any $i_0 + 1 < k < j_1$, $\#Z_k = v^{2\sum_{j \geq p_1} a_{i_k,j}}$. Then we get the coefficient of f_1 .

Finally, we fix U_k , $i_0 \leq k \leq j_1 - 1$. We need to count how many V''_{j_1} exist. Consider the set Z_{j_1} of all subspace U_{j_1} of V such that $U_{j_1-1} \subset U_{j_1} \subset V_{j_1}$, $V_{i_0} \cap U_{j_1} = V_{i_0} \cap U_{j_1-1} = U_{i_0}$ (since $b_{j_0,j_1} = 0$), $U_{j_1-1} + V_{j_1-1} \cap V'_{p_2-1} \subset U_{j_1}$, $(U_{j_1-1} + V_{j_1-1} \cap V'_{p_2}) \cap U_{j_1} \neq U_{j_1-1} + V_{j_1-1} \cap V'_{p_2}$. By the lemma 3.1.2, $\#Z_{j_1} = q^{\sum_{j \geq p_2} a_{j1,j}} - q^{\sum_{j > p_2} a_{j1,j-1}}$.

All other Z_k ($j_1 + 1 \leq k \leq i_1 - 1$) can be counted by the similar way as above. Then (a) follows.

To prove (b), we need to compute how many f'' such that $(f, f'') \in \mathcal{O}_B$, $(f'', f') \in \mathcal{O}_A$. Assume $f'' = (V''_1 \subset \cdots \subset V''_{i_0-1} \subset V''_{i_0} \subset \cdots \subset V''_{i_1-1} \subset V''_{i_1} \subset \cdots \subset V''_n)$. For any $i \geq i_0$ or $i < i_1$, $V''_i = V_i$.

First, we need to count how many V''_{i_0-1} exist. Consider the set Z_{i_0-1} of all subspace U_{i_0-1} of V such that $V_{i_0-1} \subset U_{i_0-1} \subset V_{i_0}$, $(V_{i_0-1} + V_{i_0} \cap V'_{p_1-1}) \cap U_{i_0-1} \neq U_{i_0-1}$, and $U_{i_0-1} \subset V_{i_0-1} + V_{i_0} \cap V'_{p_1}$.

$$\begin{aligned} \#Z_{i_0-1} &= \#\{U_{i_0-1} | U_{i_0-1} \subset V_{i_0-1} + V_{i_0} \cap V'_{p_1}\} - \#\{U_{i_0-1} | U_{i_0-1} \subset V_{i_0-1} + V_{i_0} \cap V'_{p_1-1}\} \\ &= \frac{v^{2(1+\sum_{j \leq p_1} a_{j0,j})} - v^{2\sum_{j < p_1} a_{j0,j}}}{v^2 - 1}. \end{aligned}$$

Second, we fix U_{i_0-1} . Consider the set Z_{i_0-2} of all subspace U_{i_0-2} of V such that $V_{i_0-2} \subset U_{i_0-2} \subset U_{i_0-1}$, $V_{i_0-1} + U_{i_0-2} = U_{i_0-1}$ (since $c_{i_0, i_0-1} = 0$), $U_{i_0-2} \subset V_{i_0-2} + U_{i_0-1} \cap V'_{p_1}$. That is, $U_{i_0-2} \subset V_{i_0-2} + U_{i_0-1} \cap V'_{p_1}$, $U_{i_0-2} \cap (V_{i_0-2} + V_{i_0-1} \cap V'_{p_1}) \neq U_{i_0-2}$.

$$\begin{aligned} \#Z_{i_0-2} &= \#\{U_{i_0-2} | U_{i_0-2} \subset V_{i_0-2} + U_{i_0-1} \cap V'_{p_1}\} - \#\{U_{i_0-2} | U_{i_0-2} \subset V_{i_0-2} + V_{i_0-1} \cap V'_{p_1}\} \\ &= \frac{v^{2(1+\sum_{j \leq p_1} a_{j_0-1,j})} - v^{2\sum_{j \leq p_1} a_{j_0-1,j}}}{v^2-1} \\ &= v^{2\sum_{j \leq p_1} a_{j_0-1,j}}. \end{aligned}$$

Similarly, as above, for any $j_1 \leq k < j_0 - 2$, $\#Z_k = v^{2\sum_{j \leq p_1} a_{k+1,j}}$. Then we get the coefficient of f_1

Finally, we fix U_k , $j_1 \leq k \leq j_0 - 1$. We need to count how many V''_{j_1-1} exist. Consider the set Z_{j_1-1} of all subspace U_{j_1-1} of V such that $V_{j_1-1} \stackrel{1}{\subset} U_{j_1-1} \subset U_{j_1}$, $U_{j_1} = V_{j_1} + U_{j_1-1}$, $U_{j_1-1} \subset V_{j_1-1} + U_{j_1} \cap V'_{p_2}$, $U_{j_1-1} \stackrel{\bullet}{\subset} V_{j_1-1} + U_{j_1} \cap V'_{p_2-1}$, where $U_{j_1-1} \stackrel{\bullet}{\subset} V_{j_1-1} + U_{j_1} \cap V'_{p_2-1}$ means $U_{j_1-1} \subset V_{j_1-1} + U_{j_1} \cap V'_{p_2-1}$ and $U_{j_1-1} \cap (V_{j_1-1} + V_{j_1} \cap V'_{p_2-1}) \neq U_{j_1-1}$.

$$\begin{aligned} \#Z_{j_1-1} &= \frac{q^{(1+\sum_{k \leq p_2})-1}}{q-1} - \frac{q^{\sum_{k < p_2}-1}}{q-1} - \left(\frac{q^{\sum_{k \leq p_2}-1}}{q-1} - \frac{q^{(\sum_{k < p_2}-1)-1}}{q-1} \right) \\ &= q^{\sum_{k \leq p_2}} - q^{(\sum_{k < p_2}-1)}. \end{aligned}$$

All other Z_k ($j_m \leq k \leq j_1 - 2$) can be counted by the similar way as above. Then (b) follows. \square

Proposition 3.1.4. *Let $A = (a_{ij}) \in \Theta_d$. Assume $B = (b_{ij}) \in \Theta_d$. There exist $1 \leq i_0 < i_1 < \dots < i_{m-1} < i_m \leq n$ such that $B - \sum_{k=1}^m E_{i_{k-1}, i_k}$ is the diagonal matrices, and $\sum_i b_{ij} = \sum_k a_{jk}$. Then*

$$e_B * e_A = \sum_{(j,p)} f(j,p) e^{(A + \sum_{1 \leq k \leq m, 1 \leq l \leq r_k} (E_{j_{k,l-1}, p_{k,l}} - E_{j_{k,l}, p_{k,l}}))},$$

where (j,p) runs over $((j_1, p_1), \dots, (j_m, p_m))$. For any $1 \leq k \leq m$,

$$(j_k, p_k) = ((j_{k,1}, p_{k,1}), \dots, (j_{k,r_k}, p_{k,r_k}))$$

satisfied the conditions: $i_{k-1} = j_{k,0} < j_{k,1} < \dots < j_{k,r_k} = i_k$, $1 \leq p_{k,r_k} < \dots < p_{k,1} \leq n$, and for any $1 \leq l \leq r_k$, $a_{j_{k,l}, p_{k,l}} \geq 1$. $f(j,p) = \prod_{1 \leq k \leq m, 1 \leq l \leq r_k} f_{k,l}$.

$$f_{k,l} = \begin{cases} \frac{v^{2(1+\sum_{j \geq p_{k,1}} a_{j_{k,0},j})} - v^{2\sum_{j \geq p_{k,1}} a_{j_{k,0},j}}}{v^2-1} \prod_{\substack{\xi=j_{k,1}-1 \\ \xi=j_{k,0}+1}}^{\xi=j_{k,1}-1} v^{2\sum_{j \geq p_{k,1}} a_{\xi,j}} & \text{if } l=1, p_{k-1, r_{k-1}} < p_{k,1}; \\ \frac{v^{2(\sum_{j \geq p_{k,1}} a_{j_{k,0},j})} - v^{2(-1+\sum_{j > p_{k,1}} a_{j_{k,0},j})}}{v^2-1} \prod_{\substack{\xi=j_{k,1}-1 \\ \xi=j_{k,0}+1}}^{\xi=j_{k,1}-1} v^{2\sum_{j \geq p_{k,1}} a_{\xi,j}} & \text{if } l=1, p_{k-1, r_{k-1}} > p_{k,1}; \\ \frac{v^{2(\sum_{j \geq p_{k,1}} a_{j_{k,0},j})} - v^{2(\sum_{j > p_{k,1}} a_{j_{k,0},j})}}{v^2-1} \prod_{\substack{\xi=j_{k,1}-1 \\ \xi=j_{k,0}+1}}^{\xi=j_{k,1}-1} v^{2\sum_{j \geq p_{k,1}} a_{\xi,j}} & \text{if } l=1, p_{k-1, r_{k-1}} = p_{k,1}; \\ (v^{2\sum_{j \geq p_{k,l}} a_{j_{k,l-1},j}} - v^{2\sum_{j > p_{k,l}} a_{j_{k,l-1},j}}) \prod_{\xi=j_{k,l-1}+1}^{k=j_{k,l}-1} v^{2\sum_{j \geq p_{k,l}} a_{\xi,j}} & \text{if } l > 1. \end{cases}$$

Proof. Assume $A' = A + \sum_{\substack{1 \leq k \leq m \\ 1 \leq l \leq r_k}} (E_{j_{k,l-1}, p_{k,l}} - E_{j_{k,l}, p_{k,l}})$. Let $f = (V_1 \subset \dots \subset V_{i_0-1} \subset V_{i_0} \subset \dots \subset$

$V_{i_m-1} \subset V_{i_m} \subset \dots \subset V_n)$, $f' = (V'_1 \subset \dots \subset V'_{i_0-1} \subset V'_{i_0} \subset \dots \subset V'_{i_m-1} \subset V'_{i_m} \subset \dots \subset V'_n)$ be such that $(f, f') \in \mathcal{O}_{A'}$. Set $V_0 = V'_0 = 0$. We need to compute how many f'' we have such that $(f, f'') \in \mathcal{O}_B$, $(f'', f') \in \mathcal{O}_A$. Assume $f'' = (V''_1 \subset \dots \subset V''_{i_0-1} \subset V''_{i_0} \subset \dots \subset V''_{i_m-1} \subset V''_{i_m} \subset \dots \subset V''_n)$. For any $i < i_0$ or $i \geq i_m$, $V''_i = V_i$. The proof of this proposition is almost

similar to the lemma 3.1.3. The only difference is that when $k > 1$, $l = 1$, how many $V''_{jk,0}$ exist. We need to count it in the following three case.

First case. $p_{k-1,r_{k-1}} < p_{k,1}$. Consider the set $Z_{jk,0}$ of all subspace $U_{jk,0}$ of V such that $V_{jk,0-1} \subset U_{jk,0} \subset V_{jk,0}$, $U_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}-1} \subset U_{jk,0}$, and $(U_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}}) \cap U_{jk,0} \neq U_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}}$.

$$\begin{aligned} \#Z_{jk,0} &= \#\{U_{jk,0} | U_{jk,0} \supset V_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}-1}\} - \#\{U_{jk,0} | U_{jk,0} \supset V_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}}\} \\ &= \frac{v^{2(1+\sum_{j \geq p_{k,1}} a_{jk,0,j})} - v^{2\sum_{j > p_{k,1}} a_{jk,0,j}}}{v^2 - 1}. \end{aligned}$$

Second case. $p_{k-1,r_{k-1}} = p_{k,1}$. Consider the set $Z'_{jk,0}$ of all subspace $U'_{jk,0}$ of V such that $V_{jk,0-1} \subset U'_{jk,0} \subset V_{jk,0}$, $U_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}-1} \subset U'_{jk,0}$, and $(U_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}}) \cap U'_{jk,0} \neq U_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}}$.

$$\begin{aligned} \#Z'_{jk,0} &= \#\{U'_{jk,0} | U'_{jk,0} \supset V_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}-1}\} - \#\{U'_{jk,0} | U'_{jk,0} \supset V_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}}\} \\ &= \frac{v^{2(\sum_{j \geq p_{k,1}} a_{jk,0,j})} - v^{2\sum_{j > p_{k,1}} a_{jk,0,j}}}{v^2 - 1}. \end{aligned}$$

Third case. $p_{k-1,r_{k-1}} > p_{k,1}$. Consider the set $Z''_{jk,0}$ of all subspace $U''_{jk,0}$ of V such that $V_{jk,0-1} \subset U''_{jk,0} \subset V_{jk,0}$, $U_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}-1} \subset U''_{jk,0}$, and $(U_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}}) \cap U''_{jk,0} \neq U_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}}$.

$$\begin{aligned} \#Z''_{jk,0} &= \#\{U''_{jk,0} | U''_{jk,0} \supset V_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}-1}\} - \#\{U''_{jk,0} | U''_{jk,0} \supset V_{jk,0-1} + V_{jk,0} \cap V'_{p_{k,1}}\} \\ &= \frac{v^{2(\sum_{j \geq p_{k,1}} a_{jk,0,j})} - v^{2\sum_{j > p_{k,1}} (-1+a_{jk,0,j})}}{v^2 - 1}. \end{aligned}$$

Then the proposition follows. \square

We assume that the ground field is an algebraic closure $\overline{\mathbb{F}}_q$ of \mathbb{F}_q when we talk about the dimension of a G -orbit or its stabilizer. Set

$$d(A) = \dim \mathcal{O}_A \quad \text{and} \quad r(A) = \dim \mathcal{O}_B, \quad \forall A \in \Theta_d,$$

where $B = (b_{ij})$ is the diagonal matrix such that $b_{ii} = \sum_k a_{ik}$. Denote by $C_G(V, V')$ the stabilizer of (V, V') in G . From [1], we have known the following fact. If $A \in \Theta_d$, we have

$$\begin{aligned} \dim C_G(V, V') &= \sum_{i \geq k, j \geq l} a_{ij} a_{kl}, \quad \text{if } (V, V') \in \mathcal{O}_A, \\ \dim \mathcal{O}_A &= \sum_{i < k \text{ or } j < l} a_{ij} a_{kl}, \\ d(A) - r(A) &= \sum_{i \geq k, j < l} a_{ij} a_{kl}. \end{aligned}$$

For any $A \in \Theta_d$, let $[A] = v^{-(d(A)-r(A))} e_A$, where e_A stand for the characteristic function of G -obits associated to A .

Define

$$\bar{t}_{ji} = 0, \quad \text{if } 1 \leq i < j \leq n,$$

$$\begin{aligned}\bar{t}_{ij} &= -(v^{-1} - v) \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-1}} v^{\lambda_i} [D_\lambda + E_{ij}] \text{ if } 1 \leq i < j \leq n, \\ \bar{t}_{ii} &= \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d}} v^{\lambda_i} [D_\lambda] \text{ if } 1 \leq i \leq n.\end{aligned}$$

Proposition 3.1.5. *The functions t_{ij} , \bar{t}_{ij} in \mathcal{S} , for any $i, j \in [1, n]$, satisfy the following relations.*

$$\begin{aligned}(R1) \quad & v^{-\delta_{ij}} \bar{t}_{ia} \bar{t}_{jb} - v^{-\delta_{ab}} \bar{t}_{jb} \bar{t}_{ia} = (v^{-1} - v)(\delta_{b < a} - \delta_{i < j}) \bar{t}_{ja} \bar{t}_{ib}, \\ (R2) \quad & \prod_{i=1}^n \bar{t}_{ii} = v^d, \\ (R3) \quad & \prod_{l=0}^d (\bar{t}_{ii} - v^l) = 0, \quad \forall i \in [1, n]. \\ (R4) \quad & \bar{t}_{ij}^{d+1} = 0, \text{ if } i \neq j.\end{aligned}$$

Proof. We show the identity R1.

We will show it in the following several cases. By the first part of lemma 3.1.3, the following identities can be obtained by directly computing.

First case. $1 \leq i < j < b < a \leq n$,

$$\begin{aligned}\bar{t}_{ia} \bar{t}_{jb} &= (v - v^{-1})^2 \sum_{\lambda} v^{\lambda_i} [D_\lambda + E_{ia}] \sum_{\lambda'} v^{\lambda_j} [D_\lambda + E_{jb}] \\ &= (v - v^{-1})^2 \sum_{\lambda} v^{-\sum_{k=i+1}^{a-1} \lambda_k} e_{D_\lambda + E_{ia}} \sum_{\lambda'} v^{-\sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda'} + E_{jb}} \\ &= (v - v^{-1})^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k - 1} e_{D_\lambda + E_{ia} + E_{jb}}. \\ \bar{t}_{jb} \bar{t}_{ia} &= (v - v^{-1})^2 \sum_{\lambda'} v^{\lambda'_j} [D_\lambda + E_{jb}] \sum_{\lambda} v^{\lambda_i} [D_\lambda + E_{ia}] \\ &= (v - v^{-1})^2 \sum_{\lambda'} v^{-\sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda'} + E_{jb}} \sum_{\lambda} v^{-\sum_{k=i+1}^{a-1} \lambda_k} e_{D_\lambda + E_{ia}} \\ &= (v - v^{-1})^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k - 1} e_{D_\lambda + E_{ia} + E_{jb}}.\end{aligned}$$

Hence

$$\bar{t}_{ia} \bar{t}_{jb} - \bar{t}_{jb} \bar{t}_{ia} = 0.$$

Second case. $i < j$ and $a \leq b$,

$$\begin{aligned}& \bar{t}_{ia} \bar{t}_{jb} \\ &= (v^{-1} - v)^2 \sum_{\lambda} v^{\lambda_i} [D_\lambda + E_{ia}] \sum_{\lambda'} v^{\lambda_j} [D_\lambda + E_{jb}] \\ &= (v^{-1} - v)^2 \sum_{\lambda} v^{-\sum_{k=i+1}^{a-1} \lambda_k} e_{D_\lambda + E_{ia}} \sum_{\lambda'} v^{-\sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda'} + E_{jb}}\end{aligned}$$

$$\begin{aligned}
&= \left\{ \begin{aligned} &(v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k + 1} e_{D_\lambda + E_{ia} + E_{jb}} && \text{if } i < j < a = b; \\ &(v^{-1} - v)^2 \left(\sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_\lambda + E_{ia} + E_{jb}} \right. \\ &+ \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k - 2} (v^2 - 1) e_{D_\lambda + E_{ib} + E_{ja}} \Big) && \text{if } i < j < a < b; \\ &(v^{-1} - v)^2 \left(\sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_\lambda + E_{ia} + E_{jb}} \right. \\ &+ \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-1}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_\lambda + E_{ib}} \Big) && \text{if } i < j = a < b; \\ &(v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_\lambda + E_{ia} + E_{jb}} && \text{if } i < a < j < b. \end{aligned} \right. \\
&\quad \bar{t}_{jb} \bar{t}_{ia} \\
&= (v^{-1} - v)^2 \sum_{\lambda'} v^{\lambda_j} [D_\lambda + E_{jb}] \sum_{\lambda} v^{\lambda_i} [D_\lambda + E_{ia}] \\
&= (v^{-1} - v)^2 \sum_{\lambda'} v^{-\sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda'} + E_{ia}} \sum_{\lambda} v^{-\sum_{k=i+1}^{a-1} \lambda_k} e_{D_\lambda + E_{ia}} \\
&= (v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_\lambda + E_{ia} + E_{jb}}.
\end{aligned}$$

$$\bar{t}_{ja} \bar{t}_{ib} = \left\{ \begin{aligned} &(v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_\lambda + E_{ib} + E_{ja}} && \text{if } i < j < a = b; \\ &(v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k - 1} e_{D_\lambda + E_{ib} + E_{ja}} && \text{if } i < j < a < b; \\ &-(v^{-1} - v) \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-1}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_\lambda + E_{ib}} && \text{if } i < j = a < b; \\ &0 && \text{if } i < a < j < b. \end{aligned} \right.$$

Thus,

$$\bar{t}_{ia} \bar{t}_{jb} - v^{-\delta_{ab}} \bar{t}_{jb} \bar{t}_{ia} = -(v^{-1} - v) \bar{t}_{ja} \bar{t}_{ib}.$$

Third case. $b < a$ and $i \geq j$,

$$\begin{aligned}
&\bar{t}_{ia} \bar{t}_{jb} \\
&= (v^{-1} - v)^2 \sum_{\lambda} v^{-\sum_{k=i+1}^{a-1} \lambda_k} e_{D_\lambda + E_{ia}} \sum_{\lambda'} v^{-\sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda'} + E_{ia}} \\
&= (v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_\lambda + E_{ia} + E_{jb}}.
\end{aligned}$$

$$\begin{aligned}
\bar{t}_{jb}\bar{t}_{ia} &= (v^{-1} - v)^2 \sum_{\lambda'} v^{\lambda_j} [D_{\lambda} + E_{jb}] \sum_{\lambda} v^{\lambda_i} [D_{\lambda} + E_{ia}] \\
&= (v^{-1} - v)^2 \sum_{\lambda'} v^{-\sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda'} + E_{jb}} \sum_{\lambda} v^{-\sum_{k=i+1}^{a-1} \lambda_k} e_{D_{\lambda} + E_{ia}} \\
&= \begin{cases} (v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda} + E_{jb} + E_{ia}} & \text{if } j < b < i < a; \\ \\ (v^{-1} - v)^2 \left(\sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda} + E_{ia} + E_{jb}} \right. \\ \left. + \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-1}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda} + E_{ja}} \right) & \text{if } j < b = i < a; \\ \\ (v^{-1} - v)^2 \left(\sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda} + E_{ia} + E_{jb}} \right. \\ \left. + \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k - 2} (v^2 - 1) e_{D_{\lambda} + E_{ib} + E_{ja}} \right) & \text{if } j < i < b < a; \\ \\ (v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k + 1} e_{D_{\lambda} + E_{jb} + E_{ia}} & \text{if } j = i < b < a. \end{cases} \\
\bar{t}_{ja}\bar{t}_{ib} &= \begin{cases} 0 & \text{if } j < b < i < a; \\ \\ -(v^{-1} - v) \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-1}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda} + E_{ja}} & \text{if } j < b = i < a; \\ \\ (v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k - 1} e_{D_{\lambda} + E_{ib} + E_{ja}} & \text{if } j < i < b < a; \\ \\ (v^{-1} - v)^2 \sum_{\substack{\lambda=(\lambda_1, \lambda_2, \dots, \lambda_n) \\ s.t. \sum_{k=1}^n \lambda_k = d-2}} v^{-\sum_{k=i+1}^{a-1} \lambda_k - \sum_{k=j+1}^{b-1} \lambda_k} e_{D_{\lambda} + E_{ja} + E_{ib}} & \text{if } j = i < b < a. \end{cases}
\end{aligned}$$

Therefore,

$$v^{-\delta_{ij}} \bar{t}_{ia} \bar{t}_{jb} - \bar{t}_{jb} \bar{t}_{ia} = (v^{-1} - v) \bar{t}_{ja} \bar{t}_{ib}.$$

Forth case. $j \leq i < a \leq b$. it is easy to know that when $j = i < a = b$, the identity R1 is equal. From the first case, when $j < i < a < b$, the identity R1 is also equal. From the second and the third case. When $j < i < a = b$ and $j = i < a < b$, the identity is right.

Thus, the identity R1 is equal.

The other identities can be shown similarly to Proposition 4.1.1 in [17]. \square

Recall the partial order “ \leq ” on Θ_d by $A \leq B$ if $\mathcal{O}_A \subset \overline{\mathcal{O}}_B[1]$. For any $A = (a_{ij})$ and $B = (b_{ij})$ in Θ_d , we say that $A \preceq B$ if and only if the following two conditions hold.

$$(6) \quad \sum_{r \leq i, s \geq j} a_{rs} \leq \sum_{r \leq i, s \geq j} b_{rs}, \quad \forall i < j.$$

$$(7) \quad \sum_{r \geq i, s \leq j} a_{rs} \leq \sum_{r \geq i, s \leq j} b_{rs}, \quad \forall i > j.$$

The relation “ \preceq ” defines a second partial order on Θ_d .

Theorem 3.1.6. *For any $A = (a_{ij}) \in \Theta_d$. The following identity holds in \mathcal{S}*

$$\prod_{1 \leq j < i \leq n} e_{D_{ij} + a_{ij} E_{ij}} * \prod_{1 \leq i < j \leq n} e_{D_{ij} + a_{ij} E_{ij}} = \chi_A e_A + \text{lower terms},$$

where $\chi_A \in \mathcal{A} \setminus \{0\}$. The factors in the first product are taken in the following order: (i, j) comes before (i', j') if either $j < j'$ or $j = j'$, $i < i'$. The factors in the second product are taken in the following order: (i, j) comes before (i', j') if either $j > j'$ or $j = j'$, $i > i'$. The matrices $D_{i,j}$ are diagonal with entries in \mathbb{N} , which are uniquely determined.

Proof. Assume $n = 3$ so that

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

We define

$$\begin{aligned} A_1 &= \begin{pmatrix} a_{11} + a_{21} + a_{31} & a_{12} & 0 \\ 0 & a_{22} + a_{32} & 0 \\ 0 & 0 & a_{13} + a_{23} + a_{33} \end{pmatrix} \\ A_2 &= \begin{pmatrix} a_{11} + a_{21} + a_{31} + a_{12} & 0 & a_{13} \\ 0 & a_{22} + a_{32} & 0 \\ 0 & 0 & a_{23} + a_{33} \end{pmatrix} \\ A_3 &= \begin{pmatrix} a_{11} + a_{21} + a_{31} + a_{12} + a_{13} & 0 & 0 \\ 0 & a_{22} + a_{32} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \\ A_4 &= \begin{pmatrix} a_{11} + a_{21} + a_{31} + a_{12} + a_{13} & 0 & 0 \\ 0 & a_{22} + a_{23} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix} \\ A_5 &= \begin{pmatrix} a_{11} + a_{21} + a_{12} + a_{13} & 0 & 0 \\ 0 & a_{22} + a_{23} & 0 \\ a_{31} & 0 & a_{32} + a_{33} \end{pmatrix} \\ A_6 &= \begin{pmatrix} a_{11} + a_{12} + a_{13} & 0 & 0 \\ a_{21} & a_{22} + a_{23} & 0 \\ 0 & 0 & a_{31} + a_{32} + a_{33} \end{pmatrix} \end{aligned}$$

By the Lemma 3.1.3, we have

$$e_{A_6} * e_{A_5} * \cdots * e_{A_1} = \chi_A e_A + \text{lower terms}.$$

Similarly, we can prove the proposition for the general case. \square

3.2. Stabilization. Let I be the identity matrix. We set ${}_pA = A + pI$. Let $\tilde{\Theta}$ be the set of all $n \times n$ matrix with integer entries such that the entries off diagonal are ≥ 0 .

Let

$$\mathcal{K} = \text{span}_{\mathcal{A}}\{[A] \mid A \in \tilde{\Theta}\},$$

where the notation $[A]$ is a formal symbol. Let v' be a independent indeterminates, and \mathfrak{R} be the ring $\mathbb{Q}(v)[v']$.

From [1], we have known the following results.

Proposition 3.2.1. *Suppose that A_1, A_2, \dots, A_r ($r \geq 2$) are matrices in $\tilde{\Theta}$ such that $\text{co}(A_i) = \text{ro}(A_{i+1})$ for $1 \leq i \leq r-1$. There exist $Z_1, \dots, Z_m \in \tilde{\Theta}$, $G_j(v, v') \in \mathfrak{R}$ and $p_0 \in \mathbb{N}$ such that in \mathcal{S}_d for some d , we have*

$$[{}_pA_1] * [{}_pA_2] * \dots * [{}_pA_r] = \sum_{j=1}^m G_j(v, v^{-p}) [{}_pZ_j], \quad \forall p \geq p_0.$$

By specialization v' at $v' = 1$, there is a unique associative \mathcal{A} -algebra structure on \mathcal{K} without unit, where the product is given by

$$[A_1] \cdot [A_2] \cdot \dots \cdot [A_r] = \sum_{j=1}^m G_j(v, 1) [Z_j]$$

if A_1, \dots, A_r are as in Proposition 3.2.1.

Let $\hat{\mathcal{K}}$ be the $\mathbb{Q}(v)$ -vector space of all formal sum $\sum_{A \in \tilde{\Theta}} \xi_A [A]$ with $\xi_A \in \mathbb{Q}(v)$ and a locally finite property, that is, for any $\mathbf{t} \in \mathbb{Z}^n$, the sets $\{A \in \tilde{\Theta} \mid \text{ro}(A) = \mathbf{t}, \xi_A \neq 0\}$ and $\{A \in \tilde{\Theta} \mid \text{co}(A) = \mathbf{t}, \xi_A \neq 0\}$ are finite. The space $\hat{\mathcal{K}}$ becomes an associative algebra over $\mathbb{Q}(v)$ which equipped with the following multiplication:

$$\sum_{A \in \tilde{\Theta}} \xi_A [A] \cdot \sum_{B \in \tilde{\Theta}} \xi_B [B] = \sum_{A, B \in \tilde{\Theta}} \xi_A \xi_B [A] \cdot [B],$$

where the product $[A] \cdot [B]$ is taken in \mathcal{K} .

For any nonzero matrix $A \in \tilde{\Theta}$, let \hat{A} be the matrix obtained by replacing diagonal entries of A by zeroes. We set $\Theta^0 = \{\hat{A} \mid A \in \tilde{\Theta}\}$.

For any $\hat{\mathbf{a}}$ in Θ^0 and $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n$, we define

$$(8) \quad \hat{\mathbf{a}}(\mathbf{j}) = \sum_{\lambda} v^{\lambda_1 j_1 + \dots + \lambda_n j_n} [\hat{\mathbf{a}} + D_{\lambda}],$$

where the sum runs through all $\lambda = (\lambda_i) \in \mathbb{Z}^n$ such that $\hat{\mathbf{a}} + D_{\lambda} \in \tilde{\Theta}$, where D_{λ} is the diagonal matrices with diagonal entries (λ_i) .

Define

$$\begin{aligned} \bar{t}_{ji} &= 0, \text{ if } 1 \leq i < j \leq n, \\ \bar{t}_{ij} &= -(v^{-1} - v) E_{ij}(\underline{i}) \quad (1 \leq i < j \leq n), \\ \bar{t}_{ii} &= 0(\underline{i}) \quad (1 \leq i \leq n), \end{aligned}$$

where $\underline{i} \in \mathbb{N}^n$ is the vector whose i -th entry is 1 and 0 elsewhere.

Let \mathcal{U} be the subalgebra of $\hat{\mathcal{K}}$ generated by $\bar{t}_{ij}, \bar{t}_{ii}$ for all $i, j \in [1, n]$ and $\mathbf{j} \in \mathbb{Z}^n$.

Proposition 3.2.2. *The generators t_{ij}, \bar{t}_{ij} in \mathcal{U} , for any $i, j \in [1, n]$, satisfy the following relations.*

$$v^{-\delta_{ij}} \bar{t}_{ia} \bar{t}_{jb} - v^{-\delta_{ab}} \bar{t}_{jb} \bar{t}_{ia} = (v^{-1} - v)(\delta_{b < a} - \delta_{i < j}) \bar{t}_{ja} \bar{t}_{ib},$$

Proof. The following identities can be obtained by directly computing by the first part of Lemma 3.1.3 and Proposition 3.2.1.

First case. $1 \leq i < j < b < a \leq n$,

$$\begin{aligned} \bar{t}_{ia} \bar{t}_{jb} &= (v - v^{-1})^2 E_{ia}(\underline{i}) E_{jb}(\underline{j}) \\ &= (v - v^{-1})^2 \sum_{\lambda} v^{\lambda_i} [D_{\lambda} + E_{ia}] \sum_{\lambda'} v^{\lambda'_j} [D_{\lambda} + E_{jb}] \\ &= (v - v^{-1})^2 \sum_{\lambda} v^{\lambda_i + \lambda_j} [D_{\lambda} + E_{ia} + E_{jb}] \\ &= (v - v^{-1})^2 (E_{ia} + E_{jb})(\underline{i} + \underline{j}). \end{aligned}$$

$$\begin{aligned} \bar{t}_{jb} \bar{t}_{ia} &= (v - v^{-1})^2 E_{jb}(\underline{j}) E_{ia}(\underline{i}) \\ &= (v - v^{-1})^2 \sum_{\lambda'} v^{\lambda'_j} [D_{\lambda} + E_{jb}] \sum_{\lambda} v^{\lambda_i} [D_{\lambda} + E_{ia}] \\ &= (v - v^{-1})^2 \sum_{\lambda} v^{\lambda_i + \lambda_j} [D_{\lambda} + E_{ia} + E_{jb}] \\ &= (v - v^{-1})^2 (E_{ia} + E_{jb})(\underline{i} + \underline{j}). \end{aligned}$$

Hence

$$\bar{t}_{ia} \bar{t}_{jb} - \bar{t}_{jb} \bar{t}_{ia} = 0.$$

Second case. $i < j$ and $a \leq b$,

$$\begin{aligned} &\bar{t}_{ia} \bar{t}_{jb} \\ &= (v^{-1} - v)^2 \sum_{\lambda} v^{\lambda_i} [D_{\lambda} + E_{ia}] \sum_{\lambda'} v^{\lambda'_j} [D_{\lambda} + E_{jb}] \\ &= \begin{cases} (v^{-1} - v)^2 v (E_{ia} + E_{jb})(\underline{i} + \underline{j}), & \text{if } i < j < a = b; \\ (v^{-1} - v)^2 ((E_{ia} + E_{jb})(\underline{i} + \underline{j}) + (v - v^{-1})(E_{ib} + E_{ja})(\underline{i} + \underline{j})) & \text{if } i < j < a < b; \\ (v^{-1} - v)^2 ((E_{ia} + E_{jb})(\underline{i} + \underline{j}) + (E_{ib})(\underline{i} + \underline{j})) & \text{if } i < j = a < b; \\ (v^{-1} - v)^2 (E_{ia} + E_{jb})(\underline{i} + \underline{j}). & \text{if } i < a < j < b. \end{cases} \\ &\bar{t}_{jb} \bar{t}_{ia} \\ &= (v^{-1} - v)^2 \sum_{\lambda'} v^{\lambda'_j} [D_{\lambda} + E_{jb}] \sum_{\lambda} v^{\lambda_i} [D_{\lambda} + E_{ia}] \\ &= (v^{-1} - v)^2 (E_{ia} + E_{jb})(\underline{i} + \underline{j}). \end{aligned}$$

$$\bar{t}_{ja}\bar{t}_{ib} = \begin{cases} (v^{-1} - v)^2(E_{ia} + E_{jb})(\underline{i} + \underline{j}), & \text{if } i < j < a = b; \\ (v^{-1} - v)^2(E_{ib} + E_{ja})(\underline{i} + \underline{j}) & \text{if } i < j < a < b; \\ -(v^{-1} - v)E_{ib}(\underline{i} + \underline{j}) & \text{if } i < j = a < b; \\ 0 & \text{if } i < a < j < b. \end{cases}$$

Thus,

$$\bar{t}_{ia}\bar{t}_{jb} - v^{-\delta_{ab}}\bar{t}_{jb}\bar{t}_{ia} = -(v^{-1} - v)\bar{t}_{ja}\bar{t}_{ib}.$$

Third case. $b < a$ and $i \geq j$,

$$\begin{aligned} & \bar{t}_{ia}\bar{t}_{jb} \\ &= (v^{-1} - v)^2 \sum_{\lambda} v^{\lambda_i} [D_{\lambda} + E_{ia}] \sum_{\lambda'} v^{\lambda'_j} [D_{\lambda'} + E_{jb}] \\ &= (v^{-1} - v)^2 v^{\delta_{ij}} (E_{ia} + E_{jb})(\underline{i} + \underline{j}) \\ & \quad \bar{t}_{jb}\bar{t}_{ia} \\ &= (v^{-1} - v)^2 \sum_{\lambda'} v^{\lambda'_j} [D_{\lambda'} + E_{jb}] \sum_{\lambda} v^{\lambda_i} [D_{\lambda} + E_{ia}] \\ &= \begin{cases} (v^{-1} - v)^2 (E_{ia} + E_{jb})(\underline{i} + \underline{j}), & \text{if } j < b < i < a; \\ (v^{-1} - v)^2 ((E_{ia} + E_{jb})(\underline{i} + \underline{j}) + E_{ja}(\underline{i} + \underline{j})) & \text{if } j < b = i < a; \\ (v^{-1} - v)^2 ((E_{ia} + E_{jb})(\underline{i} + \underline{j}) + (v - v^{-1})(E_{ib} + E_{ja})(\underline{i} + \underline{j})) & \text{if } j < i < b < a; \\ (v^{-1} - v)^2 v^2 (E_{jb} + E_{ia})(\underline{i} + \underline{j}) & \text{if } j = i < b < a. \end{cases} \\ & \quad \bar{t}_{ja}\bar{t}_{ib} \\ &= \begin{cases} 0 & \text{if } j < b < i < a; \\ -(v^{-1} - v)E_{ja}(\underline{i} + \underline{j}) & \text{if } j < b = i < a; \\ (v^{-1} - v)^2 (E_{ib} + E_{ja})(\underline{i} + \underline{j}) & \text{if } j < i < b < a; \\ (v^{-1} - v)^2 v (E_{ja} + E_{ib})(\underline{i} + \underline{j}) & \text{if } j = i < b < a. \end{cases} \end{aligned}$$

Therefore

$$v^{-\delta_{ij}}\bar{t}_{ia}\bar{t}_{jb} - \bar{t}_{jb}\bar{t}_{ia} = (v^{-1} - v)\bar{t}_{ja}\bar{t}_{ib}.$$

Forth case. $j \leq i < a \leq b$. it is easy to know that when $j = i < a = b$, the identity is equal. From the first case, when $j < i < a < b$, the identity is also equal. From the second and the third case. When $j < i < a = b$ and $j = i < a < b$, the identity is right. Thus, The proposition follows. \square

The previous proposition shows that \mathbf{U} satisfied the defining relations of the positive part of $U_v(gl_n)$ with respect to the RTT relations. In fact, $\mathbf{U} \cong U_v(gl_n)^+$. That is, we give the realization of the $U_v(gl_n)^+$ with respect to the RTT relations.

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